Lecture 11.

Measurable functions.

Def 1. If $(X, \mathcal{M})$, $(Y, \mathcal{N})$ are measurable spaces and $f: X \to Y$, then f is <u>measurable</u> (<u>$(\mathcal{M}, \mathcal{N})$-meas.</u>) if $f^{-1}(E) \in \mathcal{M}$ whenever $E \in \mathcal{N}$.

<u>Useful observation</u>. For any function $f: X \to Y$, $\{f^{-1}(E) : E \in \mathcal{N}\}$ and $\{E \subseteq Y : f^{-1}(E) \in \mathcal{M}\}$ are σ-algebras on X and Y respectively.

$\Rightarrow$

Prop 1. Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$ be MS and $f: X \to Y$. If $\mathcal{N}$ is generated by $\mathcal{E}$, then f is measurable $\Leftrightarrow$ $f^{-1}(E) \in \mathcal{M}$, $\forall E \in \mathcal{E}$.

Convention. When the target space Y is $\mathbb{R}^n$ or $\mathbb{C}^n$, then the σ-algebra is taken to be $\mathcal{B}_Y$. But, for $f: \mathbb{R} \to \mathbb{R}^n$ or $\mathbb{C}^n$, we may choose $\mathcal{B}_{\mathbb{R}}$, $\mathcal{M}_{\mu_F}$, $\mathcal{L}$ as σ-algebra on domain space $\mathbb{R}$. We then say f is Borel, μ_F, Lebesgue (or m) measurable, respectively.

Basic properties (DIY or see Folland).

- For $(X, \mathcal{M})$, $\{(Y_\alpha, \mathcal{N}_\alpha)\}_{\alpha \in A}$ meas. spaces, $f: X \to Y = \prod_{\alpha \in A} Y_\alpha$ is meas. $\Leftrightarrow$ $f_\alpha := \pi_\alpha \circ f : X \to Y_\alpha$ is meas. for every $\alpha \in A$.
- If $f, g: X \to \mathbb{R}^n$ or $\mathbb{C}^n$ are meas., then $f+g$, fg are also.

Important properties.

Let $\overline{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. (extended reals). We can make it into a metric space by choosing some cont., strictly increasing function $A : \mathbb{R} \to [a,b] \subset\subset \mathbb{R}$ and extending it to $\overline{\mathbb{R}}$ by letting $A(\pm\infty) = \lim_{x \to \pm\infty} A(x)$.

Set $d_{\overline{\mathbb{R}}}(x,y) = |A(x) - A(y)|$.

- The Borel σ-algebra $\mathcal{B}_{\overline{\mathbb{R}}}$ is then generated by $\mathcal{E} = \{[-\infty, a), (a, \infty] : a \in \mathbb{R}\}$.

- $E \in \mathcal{B}_{\overline{\mathbb{R}}} \iff E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$.

$\underline{\text{Thm 2}}$. Let $\{f_n\}_{n=1}^{\infty}$ be seq. of $\overbrace{\text{fcns}}^{\text{meas.}}$ $f_n : X \to \overline{\mathbb{R}}$, where $(X, \mathcal{M})$ is meas. space. Then

$$\sup\{f_n(x) : n \in \mathbb{N}\}, \inf\{f_n(x) : n \in \mathbb{N}\}$$
$$\limsup_{n\to\infty} f_n(x), \quad \liminf_{n\to\infty} f_n(x)$$

are all meas. fcns $X \to \overline{\mathbb{R}}$.

$\underline{\text{Pf.}}$ Let $g_N(x) = \sup\{f_n(x) : n \geq N\}$. To check meas. of g_N, suffices to check that $g_N^{-1}([-\infty, a])$, $g_N^{-1}([a, \infty])$ are in $\mathcal{M}$. Well,

$$g_N^{-1}([-\infty, a]) = \bigcup_{n=N}^{\infty} \underbrace{f_n^{-1}([-\infty, a))}_{\in \mathcal{M}}$$

$\Rightarrow g_N^{-1}([-\infty, a]) \in \mathcal{M}$.

$\Rightarrow g_N$ meas.

Similarly, $h_N(x) = \inf\{f_n(x) : n \geq N\}$ is meas.

In particular $g_1(x) = \sup_{n\geq 1}\{f_n(x)\}$ $h_1(x) = \inf_{n\geq 1}\{f_n(x)\}$ are meas. Moreover,

$$\limsup_{n\to\infty} f_n(x) = \inf_{N\in\mathbb{N}} \sup_{k\geq N}\{f_k(x)\}$$

$$= \inf_N g_N(x).$$

and similarly for liminf. $\boxtimes$

<u>Special constructions.</u>

① If $f: X \to \overline{\mathbb{R}}$, then

$$f^+ = \max(f, 0),\ f^- = \max(-f, 0),$$

the <u>positive</u> and <u>negative</u> parts of f. Note $f^+, f^- \geq 0$ and $f = f^+ - f^-$.

② If $E \subseteq X$, the <u>indicator fcn</u> $f: X \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 0, & x \notin E \\ 1, & x \in E. \end{cases}$$